

Cauchy Problem on Non-globally Hyperbolic Spacetimes

I.Ya. Arefeva, T. Ishiwatari, I.V. Volovich

*Steklov Mathematical Institute
Gubkin St. 8, 119991 Moscow, Russia
arefeva@mi.ras.ru, takumi@mi.ras.ru
volovich@mi.ras.ru*

Abstract

Solutions of the Cauchy problem for the wave equation on a non-globally hyperbolic spacetime, which contains closed timelike curves (time machines) are considered. It is proved, that there exists a solution of the Cauchy problem, it is discontinuous and in some sense unique for arbitrary initial conditions, which are given on a hypersurface at time, that precedes the formation of closed timelike curves (CTC). If the hypersurface of initial conditions intersects the region containing CTC, then the solution of the Cauchy problem exists only for such initial conditions, that satisfy a certain requirement of self-consistency.

1 Introduction

There is a well developed theory of the Cauchy problem for hyperbolic equations on globally hyperbolic spacetimes [1] - [4]. A spacetime oriented with respect to time (i.e. a pair (M, g) , where M is a smooth manifold, g is the Lorentz metric) is called globally hyperbolic, if M is diffeomorphic to $\mathbb{R}^1 \times \Sigma$, where Σ is a Cauchy surface. This definition is equivalent to the definition of global hyperbolicity of Leray [4, 5]. Hyperbolic equations on non-globally hyperbolic spacetimes have been considerably less studied, though numerous examples of such spacetimes are described by such well known solutions of equations for gravitation fields as solutions of Gödel, Kerr, Gott and many others [5] - [7] (see [8] for the history of constructing the Gödel solution). Elementary examples of non- globally hyperbolic spacetimes are the space $S_t^1 \times \mathbb{R}_x^3$ with Minkowski metric and the anti-de Sitter space. Let us note that in those cases it is natural to examine the solutions of hyperbolic equations with finite action [9].

If a Lorentz manifold contains a closed timelike curve (time machine), then it is not globally hyperbolic. There are several papers in which simplest hyperbolic equations on non-globally hyperbolic manifolds were discussed [10] - [12]. The purpose of this work is to study the wave equations on manifolds, which contain closed timelike curves. We will consider a Minkowski plane with two slits whose edges are glued in a specific manner. The obtained manifold contains the conical points and the solution of wave equation is, in general, discontinuous. We will prove that, under natural conditions on jumps of right and left modes of the solution of wave equation, the solution of Cauchy problem uniquely exists, if given initial conditions on a line at a time, which precedes the formation of closed timelike curves (in this case, this line is a Cauchy surface). If initial conditions are given on a line, which intersects the region containing closed timelike curves (in this case, this line is not a Cauchy surface), then the solution of Cauchy problem exists only when a certain condition of self-consistency is satisfied. In this case, additional initial conditions are required for uniqueness of the solution of Cauchy problem.

Motivation of this work is related with the study of possibility of creation of "worm-holes" and mini time machines in collisions of the particles at high energy [13], see also [14].

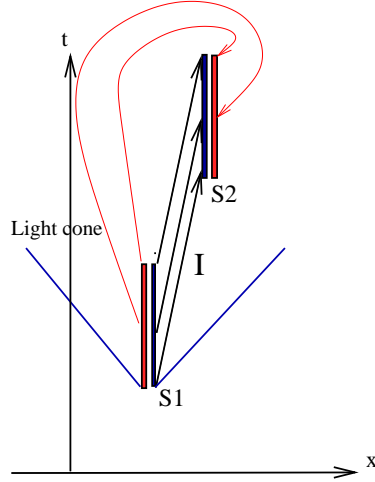


Figure 1: Minkowski plane with two segments, being glued in a specific manner : "inner" edges of the two slits are glued to each other, "outer" edges of the two slits are also glued to each other. The points of identification on "outer" edges of the two slits are indicated by lines with an arrow, the points of identification on "inner" edges of the two slits are indicated by lines with an arrow. On the picture, the light cone is drawn by lines, going out from the point S_1 with the coordinate (a_1, b_1) . It is assumed, that the vector I , which realizes the identification, is timelike and it forms a closed timelike curve. The point S_2 has the coordinate (a_2, b_2) .

2 Discontinuous solutions of Cauchy problem.

In a halfplane $\mathbb{R}_+^2 = \{(t, x) \in \mathbb{R}^2 | t > 0\}$, let us consider two vertical intervals γ_1 and γ_2 with length $l > 0$:

$$\begin{aligned}\gamma_1 &= \{(t, x) \in \mathbb{R}_+^2 \mid x = a_1, b_1 < t < b_1 + l\}, \\ \gamma_2 &= \{(t, x) \in \mathbb{R}_+^2 \mid x = a_2, b_2 < t < b_2 + l\}.\end{aligned}\tag{1}$$

We assume,

$$a_2 > a_1, \quad b_2 > b_1 + l + a_2 - a_1.$$

Suppose, that the edges of the two slits are glued as illustrated in Fig.1 [11], [12]. The obtained regions has two singular points - conical singularity - the ends of slits.

Let us consider Cauchy problem for the wave equation in an open subset (see below) of the region $\mathbb{R}_+^2 \setminus \{\overline{\gamma}_1 \cup \overline{\gamma}_2\}$ for functions $u = u(t, x)$

$$u_{tt} - u_{xx} = 0,\tag{2}$$

$$u(t, x)|_{t=0} = u_0(x), \quad \partial_t u(t, x)|_{t=0} = u_1(x),\tag{3}$$

Here $\overline{\gamma}_i$ - are closures of the intervals γ_i , $i = 1, 2$.

Suppose, function $u(t, x)$ and its first derivative with respect to t and x admit a continuous extension on the intervals γ_1 and γ_2 when (t, x) goes to γ_1 and γ_2 from right and left, and we set the following identification conditions

$$u(t, x)|_{x=a_1-0} = u(t + b_2 - b_1, x)|_{x=a_2+0}, \quad (4)$$

$$\partial_x u(t, x)|_{x=a_1-0} = \partial_x u(t + b_2 - b_1, x)|_{x=a_2+0}, \quad (5)$$

$$u(t, x)|_{x=a_1+0} = u(t + b_2 - b_1, x)|_{x=a_2-0}, \quad (6)$$

$$\partial_x u(t, x)|_{x=a_1+0} = \partial_x u(t + b_2 - b_1, x)|_{x=a_2-0}, \quad (7)$$

where $b_1 < t < b_1 + l$.

As is well-known [1], the change of variables such that

$$\tilde{u}(\xi, \eta) = u\left(\frac{\eta - \xi}{2}, \frac{\eta + \xi}{2}\right), \quad \xi = x - t, \quad \eta = x + t, \quad (8)$$

transforms the equation (2) to the canonical form

$$\frac{\partial^2 \tilde{u}(\xi, \eta)}{\partial \xi \partial \eta} = 0, \quad (9)$$

from which it follows that the solution of the equation (2) is given by

$$u(t, x) = f(x - t) + g(x + t),$$

where the functions $f(\xi)$ and $g(\eta)$ belong to the class C^2 on the corresponding intervals for variables ξ and η .

Let us consider the following problem. Suppose disconnected region $\Omega_- \subset \mathbb{R}_+^2$ is given by

$$\Omega_- = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0, \quad (10)$$

where the regions \mathcal{D}_1 , \mathcal{D}_2 and \mathcal{D}_0 are bounded by closed intervals γ_1 , γ_2 and half-lines (rays), going out from the ends of intervals γ_1 , γ_2 to the right by angle 45° (see Fig.2).

Any function w of class $C^k(\Omega_-)$ consists of three components $w = (w_0, w_1, w_2)$, where $w_i \in C^k(\mathcal{D}_i)$, $i = 0, 1, 2$.

Let us recall, that $C^k(\overline{\mathcal{D}})$ means the class of functions $C^k(\mathcal{D})$, which together with its first derivative admit a continuous extension on the closure $\overline{\mathcal{D}}$ [1].

Let us denote by $\mathcal{F}^1(\Omega_-)$ the class of such functions $w = (w_0, w_1, w_2)$ belonging to $C^1(\Omega_-)$, that $w_i \in C^1(\overline{\mathcal{D}_i})$, $i = 0, 1, 2$.

Problem Find a function $u(t, x) \in \mathcal{F}^2(\Omega_-)$, where Ω_- is a disconnected open set given by (10), which satisfies the equation

$$\frac{\partial \tilde{u}(\xi, \eta)}{\partial \eta} = 0, \quad (\xi, \eta) \in \Omega_- \quad (11)$$

(and, therefore, satisfies the wave equation (9)), also satisfies the identification conditions (4) - (7) and the initial condition such that

$$u(t, x)|_{t=0} = u_0(x), \quad x \in \mathbb{R}. \quad (12)$$

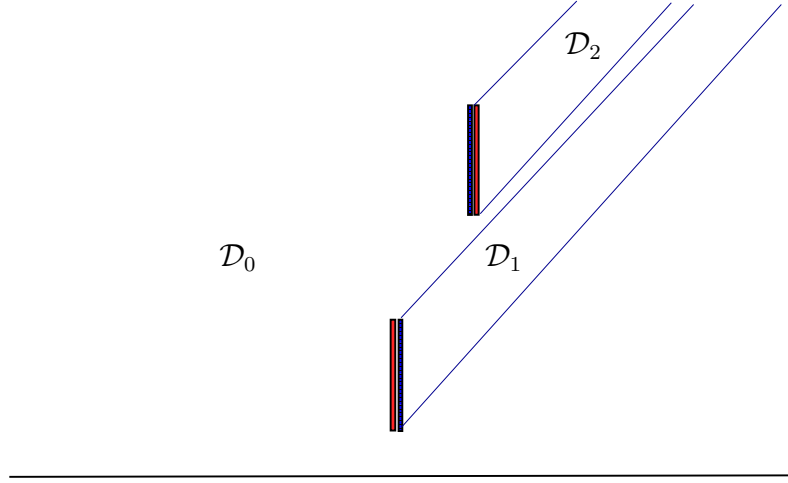


Figure 2: Region $\Omega_- = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_0$

Proposition

Let $u_0 \in C^2(\mathbb{R})$, then the solution of the problem above, belonging to the class $\mathcal{F}^2(\Omega_-)$, exists and it is unique.

Proof. From the equation (11) it follows, that in any one of regions \mathcal{D}_i , $i = 0, 1, 2$, the solution is given by the formula

$$u(t, x) = f_i(x - t), \quad (t, x) \in \mathcal{D}_i, \quad i = 0, 1, 2, \quad (13)$$

where $f_i(\xi)$ is a function belonging to the class C^2 on the region, where the argument ξ varies. From the initial condition (12) it follows

$$f_0(x) = u_0(x), \quad x \in \mathbb{R}. \quad (14)$$

Further, by using the identification condition (4), we have

$$u_0(a_1 - t) = f_2(a_2 - b_2 + b_1 - t), \quad b_1 < t < b_1 + l, \quad (15)$$

while the identification condition (6) gives

$$f_1(a_1 - t) = u_0(a_2 - b_2 + b_1 - t), \quad b_1 < t < b_1 + l. \quad (16)$$

Let us note that the identification conditions (5) and (7) for functions in the form (13) are obtained from the conditions (4) and (6), respectively.

If $(t, x) \in \mathcal{D}_1$, then $\xi = x - t$ varies in the interval (Fig.3) such that

$$a_1 - b_1 - l < \xi < a_1 - b_1, \quad (17)$$

on which the function $f_1(\xi)$ is defined.

If $(t, x) \in \mathcal{D}_2$, then $\xi = x - t$ varies in the interval such that

$$a_2 - b_2 - l < \xi < a_2 - b_2, \quad (18)$$

on which the function $f_2(\xi)$ is defined.

If $(t, x) \in D_0$, then $\xi = x - t$ varies on the whole real axis \mathbb{R} (Fig.3).

Formula (16) gives a function such that

$$f_1(\xi) = u_0(\xi + a_2 - b_2 + b_1 - a_1), \quad (19)$$

where $\xi = x - t$ varies on the interval (17).

Likewise, formula (15) defines a function such that

$$f_2(\xi) = u_0(\xi + a_1 - a_2 + b_2 - b_1), \quad (20)$$

where $\xi = x - t$ varies on the interval (18).

Thus, the solution of the problem is given by the formulas (13), where the functions f_0 , f_1 and f_2 are given by (14), (19) and (20).

Proposition is proved.

The following statement is derived from the proof of the Proposition.

Statement 1 *If the function $u = u(t, x) \in \mathcal{F}^1(\Omega_-)$ satisfies the equation (11) and the identification conditions (4) - (7), then $u(x, t)$ is given by the formulas*

$$\begin{aligned} u(t, x) &= v(x - t), & (t, x) \in \mathcal{D}_0, \\ u(t, x) &= v(x - t + a_2 - a_1 - b_2 + b_1), & (t, x) \in \mathcal{D}_1, \\ u(t, x) &= v(x - t + a_1 - a_2 + b_2 - b_1), & (t, x) \in \mathcal{D}_2, \end{aligned} \quad (21)$$

where v is some function belonging to the class $C^1(\mathbb{R})$.

Remark 1 Let us note, that some sums of jumps on the discontinuities are equal to zero. For example, $\Delta_{S_1} + \Delta_{S_2} = 0$, where $\Delta_{S_i} = u(t + 0, x) - u(t - 0, x)$ for $t - x = b_i - a_i$, $i = 1, 2$ (see Fig. 2).

A similar statement holds for $\Omega_+ = \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}'_0$, see Fig. 3.

Statement 2 *If the function $u = u(t, x) \in \mathcal{F}^1(\Omega_+)$, where $\Omega_+ = \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}'_0$ - disconnected open set shown at Fig.3, satisfies the equation*

$$\frac{\partial \tilde{u}(\xi, \eta)}{\partial \xi} = 0, \quad (\xi, \eta) \in \Omega_+, \quad (22)$$

and the conditions for identification (4) - (7), then $u(x, t)$ is given by the formulas

$$\begin{aligned} u(t, x) &= v(x + t), & (t, x) \in \mathcal{D}'_0, \\ u(t, x) &= v(x + t + a_2 - a_1 + b_2 - b_1), & (t, x) \in \mathcal{D}_3, \\ u(t, x) &= v(x + t + a_1 - a_2 + b_1 - b_2), & (t, x) \in \mathcal{D}_4, \end{aligned} \quad (23)$$

where v is some function belonging to the class $C^1(\mathbb{R})$.

The following theorem is valid.

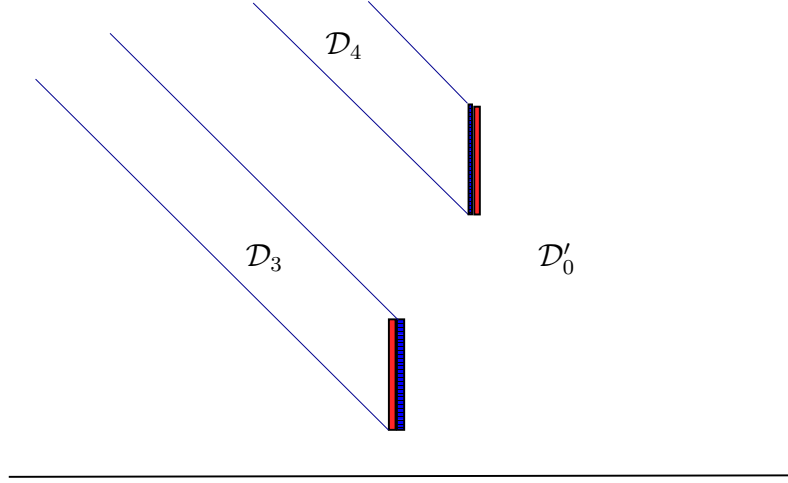


Figure 3: $\Omega_+ = \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}'_0$

Theorem 1. *Given functions $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$. Let functions f and g be defined by the following formulas*

$$f(x) = \frac{1}{2} \left[u_0(x) - \int_{x_0}^x u_1(s) ds \right], \quad g(x) = \frac{1}{2} \left[u_0(x) + \int_{x_0}^x u_1(s) ds \right], \quad (24)$$

where $x_0 \in \mathbb{R}$. Let us divide the region $\mathbb{R}_+^2 \setminus \{\bar{\gamma}_1 \cup \bar{\gamma}_2\}$ into regions \mathcal{D}_i , $i = 1, \dots, 7$ as shown in Fig. 4. Then the components of the functions $u(t, x)$, which are given in the regions \mathcal{D}_i by the following formulas

$$\begin{aligned} u(t, x) &= f(x - t + a_2 - a_1 + b_1 - b_2) + g(t + x), & (t, x) \in \mathcal{D}_1, \\ u(t, x) &= f(x - t + a_1 - a_2 + b_2 - b_1) + g(t + x), & (t, x) \in \mathcal{D}_2, \\ u(t, x) &= f(x - t) + g(t + x + a_2 - a_1 + b_2 - b_1), & (t, x) \in \mathcal{D}_3, \\ u(t, x) &= f(x - t) + g(t + x + a_1 - a_2 + b_1 - b_2), & (t, x) \in \mathcal{D}_4, \\ u(t, x) &= f(x - t) + g(t + x), & (t, x) \in \mathcal{D}_i, \quad i = 5, 6, 7, \end{aligned} \quad (25)$$

belong to the class $C^2(\mathcal{D}_i) \cap C^1(\bar{\mathcal{D}}_i)$, $i = 1, 2, \dots, 7$ and they are the solutions of the wave equation (2) in the regions \mathcal{D}_i and satisfy the initial condition (3) and the identification conditions (4) - (7).

Proof of the theorem is straightforward.

3 Minimally discontinuous solutions of wave equation

Consider now the problem on uniqueness of the solution of Cauchy problem for the wave equation on disconnected regions.

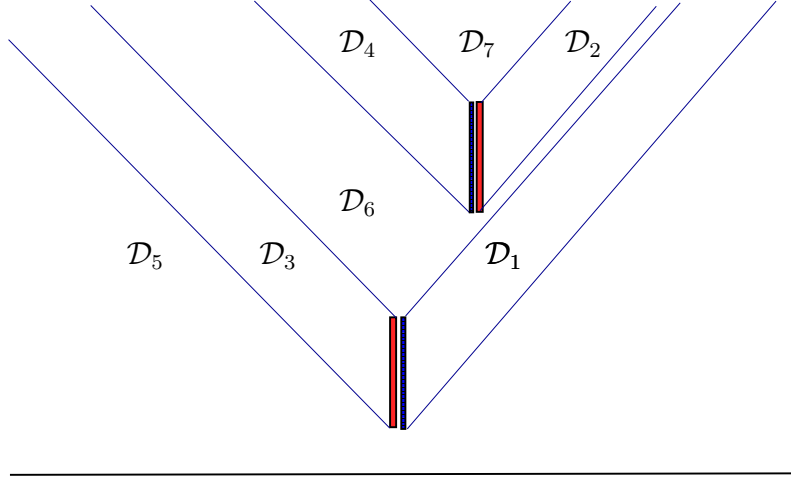


Figure 4: Regions \mathcal{D}_i , $i=1,2,\dots,7$

Suppose, for each region D_i , $i = 1, \dots, 7$, which is introduced in the Fig. 4, the solution of the wave equation belong to the class $C^2(D_i)$. Then, by the well known theorem [1], in each D_i the solution is given by a sum of the right mode and the left mode.

We say, that a function, which satisfies the wave equation in disconnected regions D_i , satisfies the condition of "minimal discontinuity", if the right and left mode, which are defined in D_i , admit the extension onto the region \mathcal{D}_0 and \mathcal{D}'_0 , respectively.

For solutions (25), let us define functions $u_r(t, x)$ (right mode);

$$\begin{aligned} u_r(t, x) &= f(x - t + a_2 - a_1 + b_1 - b_2), & (t, x) \in \mathcal{D}_1, \\ u_r(t, x) &= f(x - t + a_1 - a_2 + b_2 - b_1), & (t, x) \in \mathcal{D}_2, \\ u_r(t, x) &= f(x - t), & (t, x) \in \mathcal{D}_i, \quad i = 3, 4, 5, 6, 7, \end{aligned} \quad (26)$$

and functions $u_l(t, x)$ (left mode)

$$\begin{aligned} u_l(t, x) &= g(t + x), & (t, x) \in \mathcal{D}_i, \quad i = 1, 2, 5, 6, 7, \\ u_l(t, x) &= g(t + x + a_2 - a_1 + b_2 - b_1), & (t, x) \in \mathcal{D}_3, \\ u_l(t, x) &= g(t + x + a_1 - a_2 + b_1 - b_2), & (t, x) \in \mathcal{D}_4. \end{aligned} \quad (27)$$

Then, the function $u_r(t, x)$ (right mode), which is defined on the region \mathcal{D}_i by the formulas (26), admits a continuous extension onto the region \mathcal{D}_0 (see Fig.3), while the function $u_l(t, x)$ (left mode), which are defined in the region \mathcal{D}_i by the formulas (27), admits a continuous extension onto the region \mathcal{D}'_0 (see Fig. 3).

Theorem 2 *With the assumption of "minimal discontinuity", the solution of the problem (2) - (7) is unique and is given by the formulas (25).*

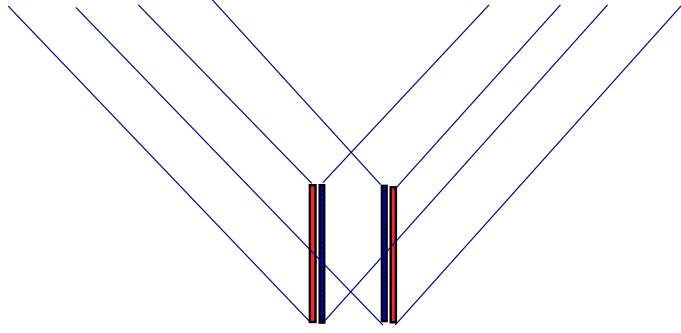


Figure 5: Two slits are at the same height and none of them is in the shadow of the other one. The vector of identification is spacelike.

Remark 2 It is possible to say, that one can consider the theorem 2 as an analogy to the edge of the wedge theorem in the theory of functions of complex variables [15].

Remark 3 It may be interesting to note that when the two slits are at the same height (Fig. 5), $b_1 = b_2$ and none of them is in the shadow of the other one, then the solutions corresponding to the problem (2), (3), (4) - (7) are given by the formulas, which are similar to (25) with $b_1 = b_2$.

The condition of "minimal discontinuity" can be formulated in the terms of the original function $u(t, x)$. We say, a solution satisfies the requirement of the condition of "minimal discontinuity" with an accuracy of jumps by constant", if the solution has left continuous derivatives (derivatives with respect to $x + t$) on the right-hand side of the light cones, the tops of which are located at the point S_i and T_i , $i = 1, 2$,

$$\partial_{t+x}u(t, x)|_{x-t=a_i-b_i+0} = \partial_{t+x}u(t, x)|_{x-t=a_i-b_i-0}, \quad t > x, \quad (28)$$

and, at the same time, the solution has also left continuous derivatives (derivatives with respect to $x - t$) on the left-hand side of the light cones, the tops of which are located at the point S_i and T_i , $i = 1, 2$,

$$\partial_{x-t}u(t, x)|_{x+t=a_i+b_i+0} = \partial_{x-t}u(t, x)|_{x+t=a_i+b_i-0}, \quad t > x. \quad (29)$$

If we assume that the right mode and the left mode decrease, then from the conditions (28) and (29) follows the minimal discontinuous condition.

4 Selfconsistent initial values of Cauchy problem

It is important to note, that for the spacetime shown in Fig. 1, it is not possible to formulate the Cauchy problem for any variable $t = t_0$. Let us recall, that the Cauchy surface in the spacetime (M, g) is such a hypersurface Σ , that every geodetic, which is released from any point of M , intersects the hypersurface Σ once and only once.

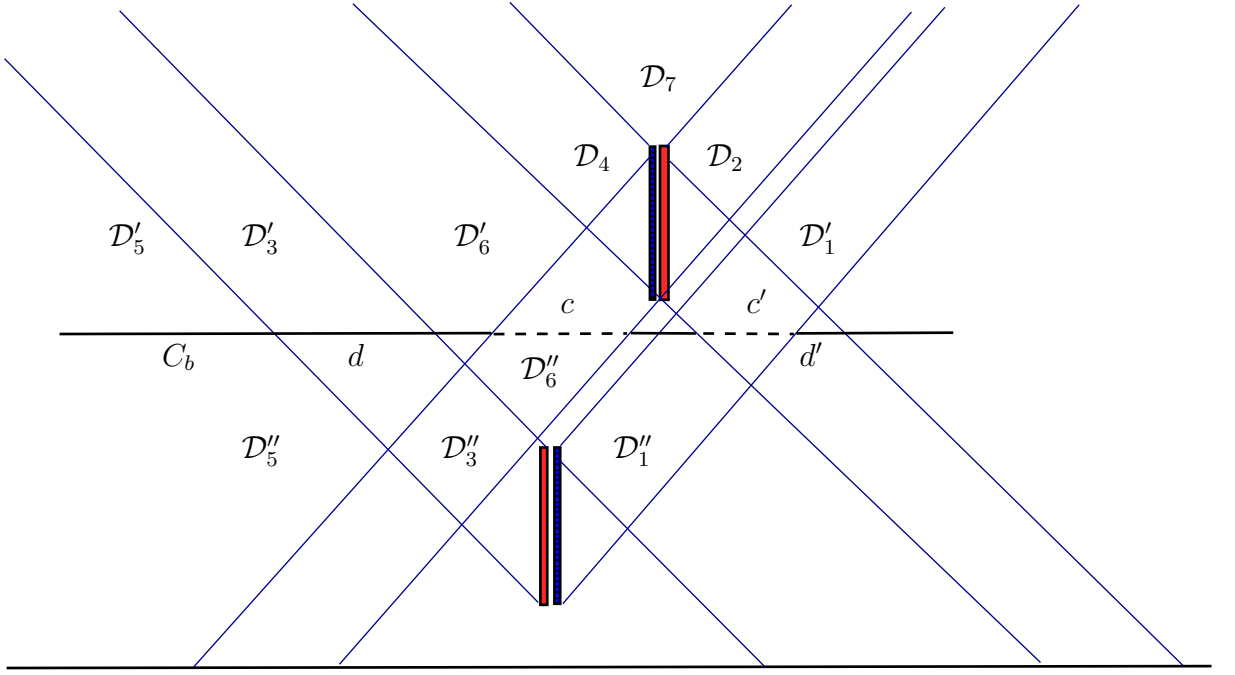


Figure 6: The surface C_b divides the region \mathcal{D}_i , $i = 1, 3, 5, 6$, illustrated in Fig. 4, into the regions \mathcal{D}'_i and \mathcal{D}''_i , $i = 1, 3, 5, 6$, respectively. It is not possible to formulate the problem (2), (3) with the identification conditions (4) - (7) and with arbitrary initial conditions on the surface C_b .

In particular, for the line C_b illustrated in Fig. 6, we can't give the right modes on the segment c independently of those on the segment c' and we can't give the left modes on the segment d independently of those on the segment d' .

This is related to the fact, that the right modes, which satisfy equation (11) and also the conditions for identification (4) - (7), do not exist for arbitrary initial conditions. The conditions for identification (4), (6) require that the initial conditions on the segments c and c' should coincide with each other, see Fig. 7. However, these initial conditions on C_b don't define the solution on the region \mathcal{D}_2 .

A similar property appears also for the wave equation with the initial conditions on the surface C_b as shown in Fig. 6.

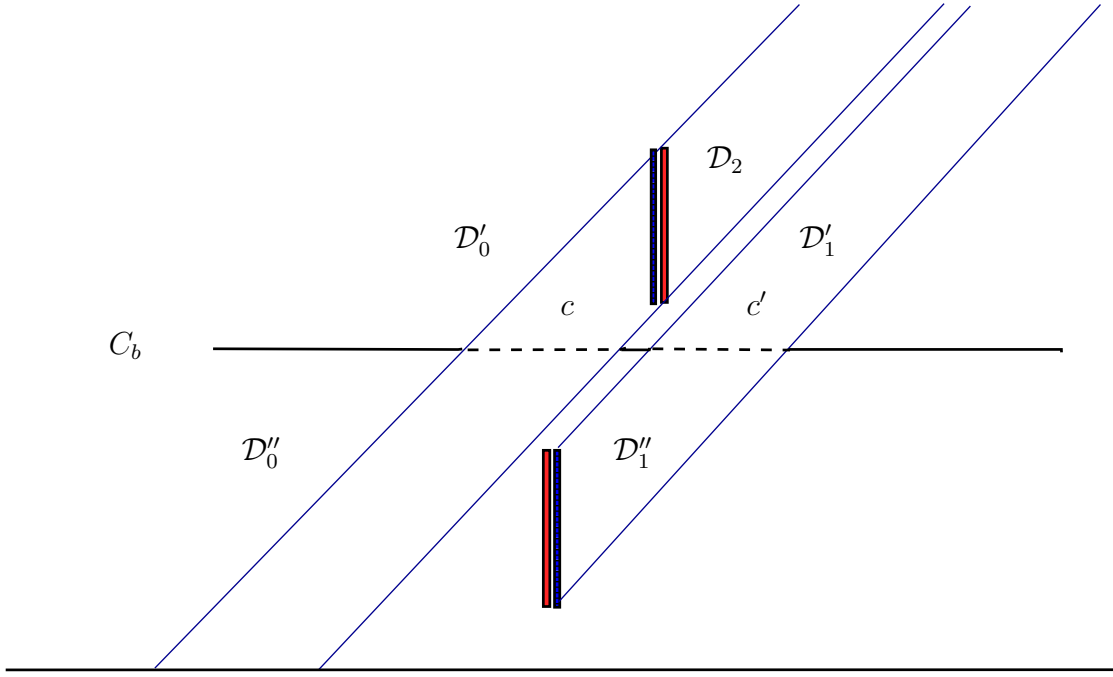


Figure 7: The surface C_b divides the region \mathcal{D}_i , $i = 0, 1$, illustrated in Fig.2, into the regions \mathcal{D}'_i and \mathcal{D}''_i , $i = 0, 1$, respectively. It is not possible to formulate the problem (11) with the identification conditions (4), (6) and with arbitrary initial conditions of the form (12) on the surface C_b .

Acknowledgements

This paper is dedicated to V.S.Vladimirov with best wishes on the 85th birthday. I.A. is partly supported by RFFI grant 08-01-00798 and grant NS-795.2008.1. I.V. is partly supported by RFFI grant 08-01-00727, NSh-3224.2008.1, DFG Project 436 RUS 113/951. T.I. is partially supported by RFFI grant 08-01-00727. I.A. would like to thank H.Nielson for fruitful discussions.

References

- [1] V.S. Vladimirov, Equations of mathematical physics, Moscow : URSS, 1996 .
- [2] J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Dover, New York, 1952 .
- [3] I.G. Petrowsky, *Über das Cauchysche Problem für Systeme von partiellen Differentialgleichungen*, Mat. Sb., Vol. 2(1937), 815–870.
- [4] J. Leray, Hyperbolic Differential Equations, Institute for Advanced Study, Princeton, N.J., 1972 .
- [5] Hawking S., Ellis J. The Large Scale Structure of Spacetime , Cambridge University Press, 1973 .
- [6] M. Visser, Lorentzian Wormholes, Springer-Verlag, 1995.
- [7] J. R. Gott, Time Travel in Einstein's Universe, Houghton Mifflin, New York, 2001.
- [8] I. Todorov, " Kurt Goedel and his universe," arXiv:0709.1387.v3 (2007).
- [9] V.V. Kozlov, I.V. Volovich, *Finite Action Klein-Gordon Solutions on Lorentzian Manifolds*, Int.J.Geom.Meth.Mod.Phys.3 (2006) 1349-1358 ; gr-qc/0603111.
- [10] J. Friedman, M.S. Morris, I.D. Novikov, F. Echeverria, G. Klinkhammer, K.S. Thorne and U. Yurtsever, *Cauchy problem in spacetimes with closed timelike curves* Phys. Rev. D**42** 1915 (1990)
- [11] D. Deutsch, *Quantum mechanics near closed timelike lines*. Phys. Rev. D**44** 3197 (1991)
- [12] H.D. Politzer, *Path integrals, density matrices, and information flow with closed timelike curves* Phys. Rev. D **49** 3981 (1994)
- [13] I.Ya. Aref'eva, I.V. Volovich, *Time Machine at the LHC*, Int.J.Geom.Meth.Mod.Phys. 5 (2008) 641-651; arXiv:07102696.
- [14] A. Mironov, A. Morozov, T.N. Tomaras, *If LHC is a Mini-Time-Machines Factory, Can We Notice?*; arXiv:0710.3395.
- [15] V.S. Vladimirov, Methods of the theory of functions of several complex variables, M.I.T., 1966 .